Emittance growth of bunched beams in bends

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Talman [Phys. Rev. Lett. **56**, 1429 (1986)] has proposed a novel relativistic effect that occurs when a charged particle beam is bent in the magnetic field from an external dipole. The consequence of this effect is that the space-charge forces from the particles do not exhibit the usual inverse-square energy dependence and some part of them are, in fact, independent of energy. This led to speculation that this effect could introduce significant emittance growth for a bending electron beam. Subsequently, it was shown that this effect's influence on the beam's transverse motion is canceled for a dc beam by a potential depression within the beam (to first order in the beam radius divided by the bend radius). In this paper, we extend the analysis to include short bunch lengths (as compared to the beam pipe dimensions) and find that there is no longer the cancellation for forces both transverse to and in the direction of motion. We provide an estimate for the emittance growth as a function of bend angle, beam radius, and current, and for magnetic compression of an electron bunch.

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I. INTRODUCTION

There is a trend in electron accelerators towards higher peak currents and lower transverse emittances. For example, designs for future linear colliders [1,2] and drivers for future short-wavelength free-electron lasers (FELs) [3,4] typically require beams with transverse emittances of a few μm and peak currents in excess of 1 kA. Because the transverse space-charge forces are expected to scale inversely with the square of the relativistic mass factor, these designs attain the high peak currents with magnetic bunch compression at relatively high energies.

Talman [5] and later Piwinski [6] described a space-charge force, known as the centrifugal-space-charge force, that does not exhibit the relativistic cancellation if the beam path is circular; however, others [7,8] pointed out that the effect is canceled by the variation in beam energy introduced by the potential depression of the beam to first order in the beam radius divided by the bend radius. The force that Talman discussed originated from the dependence of the radiation terms on the charged particles' acceleration, and is a dc phenomenon. Piwinski examined the force for a harmonic current variation in a rectangular beam pipe but only in the limit where the periodicity is long in comparison to the beam pipe dimensions.

In this paper, we discuss additional force terms for a bunch in a bend which are not canceled by the effect of the beam's depression. These terms are related to Talman's and Piwinski's effects but arise when the beam current periodicity (or overall bunch length) is short in comparison to the beam pipe aperture. In particular, we examine space-charge forces both in the transverse direction and in the direction of motion that do not exhibit the usual relativistic cancellation. We assume that the bend radius is very large and in general only keep the lowest order term in factors of the beam radius divided by the bend radius.

We find that if the beam radius is large (on the order of 1 mm) an appreciable emittance growth can occur from both the transverse force and the energy spread induced from the electric field in the direction of motion. This emittance growth can easily exceed 10 μm for beams of 1 kA or more with bunch lengths on the order of 1 ps.

We calculate this effect by finding the space-charge forces of a bunch while in a bend. The current of a nominal-shape bunch will be harmonically decomposed and broken into separate lines of charge. The space-charge fields for each harmonic are calculated by a perturbation analysis of the scalar and vector potentials using the wave equation and the Lorentz gauge condition. The forces transverse to and along the direction of motion are calculated by explicit expansion of the Lorentz force in terms of the potentials. The harmonics are summed up and the fields are numerically integrated over a uniform beam cross section.

The space-charge force along the direction of motion changes the rms energy spread of the beam. Because of the form of the space-charge forces, the emittance growth can be estimated from this energy spread. We show that the transverse space-charge force also leads to an effective energy spread and estimate the emittance growth from it. These calculations provide an estimate for the magnitude of the potential emittance growth and show how this emittance growth scales with different beam parameters. The actual emittance growth is dependent on the precise beam distribution function and correlations within the beam. Certain correlations can lead to partial or even total cancellation of the emittance growth. However, the beam distribution will vary within a bend and such cancellation cannot be expected. Thus the goals of these calculations are to provide guidelines for when the potential emittance growth becomes significant and scaling laws which indicate how to reduce it.

This paper is divided into six sections which will address different aspects of the problem. In the first section, we review the "classical" emittance growth of a bunched beam in both a drift and in a bend, using the usual space-charge forces which are inversely proportional to the square of the beam energy. In the second section, we calculate the centrifugal-space-charge force of a dc beam in a bend and show, as expected, that the effect of the space-charge potential depression cancels the effect from the centrifugal-space-charge force. This is an important result because it shows that the part of the vector potential that can be written as $\mathbf{A}=(0,(\beta/c)\phi,0)$, where ϕ is the scalar potential, has no effect on the beam dynamics; we use this fact later as the basis for our perturbation analysis.

In the third section we do an explicit derivation of the space-charge fields around an infinite rotating cylinder of charge. Some details of this tedious calculation are provided in the Appendix. Although this geometry is not of practical interest, the results show that (1) the space-charge fields (to first order in the inverse bend radius) are the same as the ones found in the straight-line motion for bunch lengths greater than a certain value (depending on the bend radius and the beam radius) and (2) the space-charge fields are higher order and thus much smaller for shorter bunch lengths. In current accelerator technology and in projected future machines, all bunch lengths that are feasible fall under case (1). Thus we will use straight-line fields as the initial estimate in the perturbation analysis. In the fourth section we do the perturbation analysis. We start with the wave equations and the Lorentz gauge condition and assume the vector potential is of the form $\mathbf{A}_n=(\delta A_{r,n},(\beta/c)\phi_n+\delta A_{\theta,n},0)$ for each harmonic n and a thin ring of current. We get equations for $\delta A_{r,n}$ and $\delta A_{\theta,n}$ in terms of ϕ_n, and then find explicit expressions using the linear motion expressions for ϕ_n.

In the fifth section, we write the transverse Lorentz force equation in terms of the perturbed potentials. Now the advantage of this approach becomes clear—all terms first order in ϕ_n cancel, leaving only the smaller perturbed fields. Thus even if the error in the field assumption is large compared to the perturbed potentials, the result is accurate as long as the error is small compared to ϕ_n. In the transverse equation of motion, two types of terms arise that are energy independent and were not present in the dc case. To relate this result to previous work, we then look at Piwinski's results in both the long beam and the short beam limits and show that his results are consistent with the appearance of these terms as the bunch length becomes small.

In the sixth section, we estimate the emittance growth from these two new terms. Each term is treated separately because they arise from fundamentally different effects. First, there is a term corresponding to the net centrifugal-space-charge force, which is summed over all harmonic contributions. An effective energy spread is found for this force and the emittance growth is estimated from this energy spread. The emittance growth is independent of beam energy and linear with bend angle, but scales inversely with the bunch length and the square root of the bend radius and roughly to the $\frac{5}{2}$ power of the beam radius. This emittance growth is typically small (<1 μm) but can become large for large-radius short (subpicosecond) bunches in relatively small-radius bends (on the order of 1 m). The second term leads to an energy spread increase from the azimuthal electric field as the beam drifts, and also can be expressed as an emittance growth. The emittance growth from this effect is quadratic with bend angle and beam radius and inversely proportional to the bunch length. For short bunches with large radius this emittance growth can be extremely large (greater than 100 μm) for bends of 1 rad or more.

II. EMITTANCE GROWTH FROM THE USUAL LINEAR-MOTION SPACE-CHARGE FIELDS

In this section we discuss the emittance growth from the "classical" space-charge forces which depend inversely on the square of the beam energy. The purpose of this section is to introduce some of the concepts of emittance growth and to provide a formalism to describe how an energy spread in the beam results in an emittance growth. The rms transverse normalized emittance (in the x direction) is defined by

$$\varepsilon_n=\beta\gamma\sqrt{\langle x^2\rangle\langle x'^2\rangle-\langle xx'\rangle^2}\ , \tag{1}$$

where β and γ are the usual relativistic factors, the brackets imply ensemble averages, and the prime indicates an axial derivative. The normalized rms emittance growth of a drifting slug beam has been studied in [9]. The emittance growth is linear with distance as long as the beam radius does not significantly deform; thereafter the emittance growth is less than linear with distance. In the linear emittance growth regime the beam has not appreciably deformed radially and the emittance growth can be thought of as the rms change in the beam divergence from the space-charge forces times the beam rms radius. In this case the normalized rms emittance growth is given by [9]

$$\Delta\varepsilon_n=\frac{IS}{4I_A\beta^2\gamma^2}\chi\ , \tag{2}$$

where I is the beam current,

$$I_A=\frac{4\pi\varepsilon mc^3}{e}=17\ \text{kA}\ , \tag{3}$$

S is the path length, and χ is a geometric factor typically between 0.2 and 0.5. This emittance growth, like all others considered in this paper, adds in quadrature to the initial emittance:

$$\varepsilon_f^2=\varepsilon_i^2+\Delta\varepsilon_n^2\ , \tag{4}$$

where ε_f is the final normalized emittance and ε_i is the initial normalized emittance. The actual increase in the emittance is given by

$$\varepsilon_f-\varepsilon_i=\frac{1}{2}\frac{\Delta\varepsilon_n^2}{\varepsilon_i},\quad \varepsilon_i \gg \Delta\varepsilon_n \ , \tag{5}$$

if the initial emittance is much larger than the emittance growth and given by

$$\varepsilon_f-\varepsilon_i=\Delta\varepsilon_n,\quad \varepsilon_i \ll \Delta\varepsilon_n \ , \tag{6}$$

if the emittance growth is much larger than the initial emittance. Note that the *linear* increase in the emittance is smaller than $\Delta\varepsilon_n$ if the initial emittance is not negligible. Defining the emittance growth as the in-quadrature component allows us to talk about an "emittance growth" which does not depend on the initial emittance.

It is easy to show that the radial beam expansion is approximately the normalized rms emittance growth [Eq. (2)] times the drift distance divided by the beam radius and γ. Since we are considering emittance growths on the order of a μm for ultrarelativistic energies, the beam's radial expansion is not significant until the beam has drifted several hundred meters or even kilometers. Thus we can in general assume that the emittance growth within bend systems (typically with path lengths of only a few meters) is dominated by the change in the rms beam divergence without deforming the charge distribution within the bunch. This is an important observation and we will use this fact to estimate emittance growths later in this paper.

In addition to the drifting-beam emittance growth, there are additional emittance growth mechanisms if the beam is bent in a dipole magnetic field. Bend systems in low-emittance accelerators are usually designed to be achromatic; that is, there is no emittance growth if the beam is not monoenergetic. However, these designs do not compensate for energy changes within the bend system itself. Consider a two-dipole bend, which is made achromatic by including a quadrupole between the dipoles. If a particle's energy changes when it is between the dipoles (for example, from the space-charge force along the direction of motion), the final bend angle will be modified by

$$\Delta\alpha=\frac{\Delta E}{E}\alpha \ , \tag{7}$$

where ΔE is the energy change, E is its initial energy, and α is the nominal bend angle from the final dipole. Using Eq. (1) and the fact that the bend is otherwise achromatic the emittance growth for an ensemble of particles can be shown to be

$$\Delta\varepsilon_n=\gamma\beta\frac{\alpha}{E}\left[\{\langle x^2\rangle\langle(\Delta E-\Delta E_{\rm ave})^2\rangle-\langle x\Delta E\rangle^2\} +2\frac{\alpha}{E}\{\langle x^2\rangle\langle x'\Delta E\rangle-\langle xx'\rangle\langle x\Delta E\rangle\}\right]^{1/2} , \tag{8}$$

where $\Delta E_{\rm ave}$ is the average energy change. Since we are concerned with the energy changes due to space-charge forces, $\Delta E_{\rm ave}$ is zero. The energy change is often either an even function of x (for example, if the energy change is from the longitudinal space-charge force for straight-line motion) or linear with x [we will see that this is the case with the energy independent space-charge forces described later in this paper—see, for example, Eqs. (104) and (117)] times a function g that just depends on the longitudinal position within the bunch. Additionally, the particle distribution is usually separable into a radial function and an axial function.

If the energy change is an even function of x the second and fourth terms vanish. Assuming that the initial emittance is small, x' is either small or nearly linear with x, and the third term is also very small, leaving just the first term which we can identify with the rms energy change. If the energy change is linear with x the third and fourth terms cancel, and the emittance growth is given by

$$\Delta\varepsilon_n=\gamma\beta\frac{\alpha}{E}\langle x^2\rangle\{\langle g^2\rangle-\langle g\rangle^2\}^{1/2} . \tag{9}$$

It is clear that the actual emittance growth is strongly dependent on the exact axial distribution function and the correlations of the energy spread with position within the bunch. We can write the emittance change as

$$\begin{aligned}\Delta\varepsilon_n&=\alpha\frac{\beta\gamma}{2E}(a^2\Delta E_{\rm rms}^2-\langle x\Delta E\rangle^2)^{1/2}\\ &=\alpha\frac{\beta}{2mc^2}(a^2\Delta E_{\rm rms}^2-\langle x\Delta E\rangle^2)^{1/2} ,\end{aligned} \tag{10}$$

where a is twice the rms transverse beam size (and equal to the beam radius if the beam cross section is a uniformly filled circle) and $\Delta E_{\rm rms}$ is the rms energy change. Note that the emittance growth from an energy spread is independent of the beam's energy. The second term under the radical sign in Eq. (10) never increases the emittance growth. Thus the maximum emittance growth resulting from a rms energy spread is given by

$$\Delta\varepsilon_{n,\max}=\beta\frac{a}{2}\frac{\Delta E_{\rm rms}}{mc^2}\alpha \ , \tag{11}$$

which equals the actual emittance growth if $\langle x\Delta E\rangle$ equals zero. We can use these equations to estimate the emittance growth by calculating the growth in the rms energy spread as the beam is transported in a bend. Note that Eqs. (8)–(11) are consistent with the observation that the emittance growth is dominated by the change in the beam's radial velocities and independent of any radial deformation. If there are particle energy changes within an individual dipole, Eq. (10) written in differential form can be integrated along the dipole with respect to the bend angle to find the emittance growth.

We can estimate the emittance growth as the beam is bent by approximating the actual space-charge fields by the fields resulting from straight-line motion. It must be emphasized that this is incorrect (as we will see later), but this is a standard approximation and is used in all accelerator transport codes. Consider a bunch of length δ in a bend of radius R (Fig. 1). We can define a local frame of reference with coordinates $(x,z,R\theta)$, pictured in Fig. 2. If the bunch length is negligible in comparison to the bend radius, this local frame is an approximate

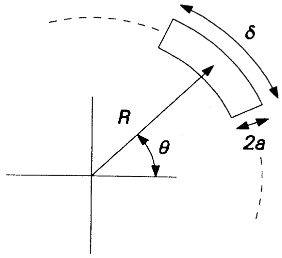

FIG. 1. Bunch of length δ and radius a in bend of radius R.

inertial frame for the bunch at that instant because the beam appears to be going in a straight line. For a sufficiently relativistic, uniform radial-density beam in straight-line motion, the radial electric field in its own instantaneous inertial frame is given by

$$E_{r,i}=\frac{\rho I}{2\pi\varepsilon a^2\beta\gamma c} \tag{12}$$

inside the beam ($\rho^2=x^2+z^2$) and

$$E_{r,o}=\frac{I}{2\pi\varepsilon\beta\gamma c\rho} \tag{13}$$

outside the beam, where I is the local current, typically a function of the azimuthal (along the direction of motion) position within the beam $R\theta$. The radially integrated electric field from a position r within the beam to the wall at radius b is given by

$$\int_r^b E_r dr=\frac{I}{2\pi\varepsilon\beta\gamma ca^2}\left[\frac{a^2-\rho^2}{2}+a^2\ln\frac{b}{a}\right] . \tag{14}$$

We can use Stokes' law and the curl equation for the electric field in the beam's own inertial frame to find the axial electric field, which is given by the derivative of this expression with respect to the axial distance. Recall, though, that the axial distance in the beam frame is given by $z\gamma$ and that the longitudinal electric field is the same in the laboratory and beam frames. This gives in the laboratory frame

$$E_z=\frac{dI}{dz}\frac{1}{2\pi\varepsilon\beta\gamma^2ca^2}\left[\frac{a^2-\rho^2}{2}+a^2\ln\frac{b}{a}\right] \tag{15}$$

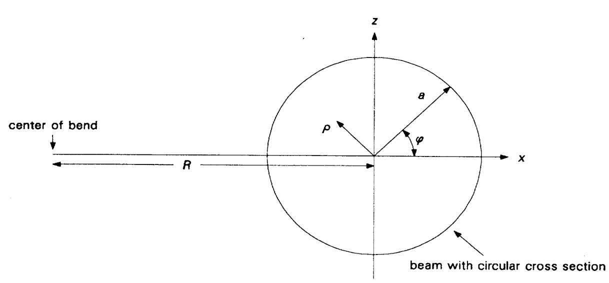

FIG. 2. Geometry used for the beam's local coordinate system. The beam is moving out of the plane of the page.

and the energy change of a particle at $(x,z,R\theta)$ after the beam drifts a distance S is given by

$$\Delta E=\frac{dI}{dz}\frac{eS}{2\pi\varepsilon\beta\gamma^2ca^2}\left[\frac{a^2-\rho^2}{2}+a^2\ln\frac{b}{a}\right] . \tag{16}$$

The rms energy change for a bunch with a parabolic current profile and peak current I_p and length δ is

$$\Delta E_{\text{rms}}=\frac{3}{4\delta}\frac{eI_pS}{4\pi\varepsilon\beta^2\gamma^2c}\left[\frac{1}{2}+2\ln\frac{b}{a}\right] . \tag{17}$$

Note that ΔE is linear with z for this case and $\langle x\Delta E\rangle$ vanishes. Thus, as the beam bends an angle $d\alpha$, the rms normalized emittance grows an amount

$$d\varepsilon_n=\gamma\beta\frac{a}{2}\frac{\Delta E_{\text{rms}}}{E}d\alpha . \tag{18}$$

Assuming either a distributed bending of the beam or just two individual dipoles leads to the same emittance growth:

$$\Delta\varepsilon_n=\alpha\frac{3a}{16\delta}\frac{I_pS}{I_A\beta\gamma^2}\left[\frac{1}{2}+2\ln\frac{b}{a}\right] . \tag{19}$$

This emittance growth can be relatively quite large [compared to the straight-line growth (Eq. (2)] for short bunch lengths. For example, a 170-A peak-current bunch 1 mm in radius and 1-ps long in a 180° bend at 10 MeV will lead to 100 μm of emittance growth per meter of bend if the beam pipe radius is 3 mm.

The strong scaling in Eqs. (2) and (19) of the emittance growth with beam energy leads to the common myth that no emittance growth results from bending an ultrarelativistic beam.

As a final exercise in this section, let us estimate the emittance growth in a nonachromatic bend for a beam that has an energy spread of the size of its space-charge potential depression. Alternatively, this is the emittance growth in a two-dipole achromatic bend in which there is some mechanism (for example, longitudinal wake fields) generating this type of additional uncorrelated energy spread while the bunch is drifting between the dipoles. The space-charge potential

$$\phi=\frac{I\rho^2}{4\pi\varepsilon ca^2} \tag{20}$$

leads to a rms energy spread of

$$\Delta E_{\text{rms}}=\frac{eI}{4\sqrt{3}\pi\varepsilon c} . \tag{21}$$

Using Eq. (11) (because $\langle x\Delta E\rangle$ equals zero), this will lead to an emittance growth

$$\Delta\varepsilon_n=\gamma\beta\frac{a}{2}\frac{\Delta E_{\text{rms}}}{E}\alpha$$

$$=\beta\frac{a}{2}\frac{eI}{4\sqrt{3}\pi\varepsilon mc^3}\alpha$$

$$=\frac{I}{I_A}\frac{a\alpha}{2\sqrt{3}} , \tag{22}$$

which, as before, is independent of the beam energy.

We will use this technique in estimating the emittance growth from different effects. It must be emphasized that this is not the emittance growth due to the potential depression of the beam for two reasons. First, if the bend is achromatic, only changes in the particle's energy would lead to an emittance growth, and second, there are centrifugal-space-charge forces which tend to cancel this effect. However, this technique can be applied to estimating the emittance growth from other forces leading to a rms energy spread.

III. dc BEAM IN BEND

There are two important points we will make in this section. First, we show that if the vector potential is equal to $\mathbf{A}=(0,(\beta/c)\phi,0)$, then the centrifugal-space-charge force cancels the effect of the potential depression of the beam (almost exactly). The next thing we show is that any noncancellation of the centrifugal-space-charge force and the potential depression looks like an energy spread, and the emittance calculation technique from the preceding section can be used to find the emittance growth.

We begin by first reviewing the case of a dc beam within a beam pipe, following [7]. The vector potential $\mathbf{A}$ and scalar potential ϕ are given by

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \tag{23}$$

and

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon} , \tag{24}$$

where $\mathbf{J}$ and ρ are the usual current and charge densities. If we define $x=r-R$ we can write, to first order in x/R (using the cylindrical coordinates in Fig. 1),

$$A_\theta = \frac{\beta}{c}\phi . \tag{25}$$

In the dc case, the time derivatives vanish and the radial force equation gives

$$F_r = e\left[-\frac{1}{\gamma^2}\frac{d\phi}{dr} + \beta^2\frac{\phi}{R}\right] . \tag{26}$$

In a later section we will calculate the higher-order correction terms for the vector potential which will in turn lead to an additional term of order $1/R^2$ in the force equation. The second term on the right-hand side of Eq. (26) is the curvature term attributed to Talman. We apply Newton's second law, which results in

$$\frac{d}{dt}(\gamma\dot{r}) = \frac{\gamma v^2}{r} - e\frac{vB_0}{m} + e\frac{\beta^2\phi}{mR} - \frac{e}{\gamma^2 m}\frac{d\phi}{dr} , \tag{27}$$

where v is the particles' velocity (close to c) and B_0 is the applied external magnetic field. We can rewrite Eq. (27) by introducing $\gamma_1 = \gamma - \gamma_0$ as

$$\begin{aligned}\gamma\ddot{x} = {} & -\dot{\gamma}_1\dot{x} + \frac{\gamma_0 v^2}{R}\left[1 - \frac{x}{R} + \frac{x^2}{2R^2} + \cdots\right] \\ & + \frac{\gamma_1 v^2}{R}\left[1 - \frac{x}{R} + \frac{x^2}{2R^2} + \cdots\right] \\ & - e\frac{vB_0}{m} + e\frac{\beta^2\phi}{mR} - \frac{e}{\gamma^2 m}\frac{d\phi}{dr} .\end{aligned} \tag{28}$$

If we assume an equilibrium orbit for a particle at the center of the bunch (where the radial derivative of the scalar potential vanishes) with relativistic mass factor γ_0, then

$$0 = \frac{\gamma_0 v^2}{R} - e\frac{vB_0}{m} + e\frac{\beta^2\phi(0,0)}{mR} , \tag{29}$$

where the scalar potential is written as a function of the local coordinates (x,z). Equation (28) reduces to

$$\begin{aligned}\ddot{x} = {} & -\frac{\dot{\gamma}_1}{\gamma}\dot{x} + \frac{v^2}{R}\left[-\frac{x}{R} + \frac{x^2}{2R^2} + \cdots\right] + \frac{\gamma_1 v^2}{\gamma R} \\ & + e\frac{\beta^2(\phi(x,z) - \phi(0,0))}{\gamma m R} - \frac{e}{\gamma^3 m}\frac{d\phi}{dr} .\end{aligned} \tag{30}$$

None of the terms depending on v^2x^n/R^{n+1} will lead to an emittance growth in an achromatic bend (they are just the usual higher-order bend optics terms) and we can ignore them. Now γ_1 depends on the position of the particle within the bunch because of energy conservation. If the variation in particle energy only depends on the space-charge forces,

$$\gamma_1 = -e\frac{\phi(x,z) - \phi(0,0)}{mc^2} . \tag{31}$$

The $\gamma_1 v^2/\gamma R$ term cancels the curvature term exactly, leaving

$$\ddot{x} = -\frac{v^2}{R^2}\left[-\frac{x}{R} + \frac{x^2}{2R^2} - \frac{x^3}{6R^3}\right] + \frac{e}{m\gamma}\frac{d\phi}{dr}\left[\beta_t^2 - \frac{1}{\gamma^2}\right] , \tag{32}$$

where β_t ($\ll 1$) is the transverse normalized velocity. Note that γ_1 now only occurs in the "normal" space-charge term. In other words, the effect of the potential depression of the beam has nearly vanished. Physically, as the beam is bent, the potential energy in the Coulomb fields around the beam must also be bent. By the conservation of energy and the equivalence of mass and energy, the inertia of the potential depression is the same as the inertia of the potential fields. We would expect the potential energy that an individual particle drags around with it should be related to the particle's potential depression, but it is not obvious that they should be equal to lowest order in $1/R$.

The error in the assumed form of the vector potential in the above analysis will lead to a noncanceling centrifugal space-charge force of order $1/R^2$ (this will become apparent in a later section). However, if the beam is bunched, there are additional errors introduced into the above analysis which will lead to terms inversely proportional to R. There are two additional physical effects introduced by a bunched beam. First, the radial component of the vector potential becomes nonzero, primarily arising from the time derivatives. Also, γ_1 depends on the azimuthal electric field, which also becomes nonzero. As a result, a more complicated analysis is required to find the particles' transverse motion.

For convenience we can rewrite the transverse acceleration including the additional forces in the form [using Eq. (29) to define the equilibrium orbit]:

$$\ddot{x}=\frac{v^2}{R}\left[\frac{x}{R}-\frac{x^2}{2R^2}+\frac{x^3}{6R^3}\right]+\frac{e}{m\gamma}\frac{d\phi}{dr}\left[\beta_t^2-\frac{1}{\gamma^2}\right]-\frac{e\beta^2}{\gamma mR}\phi(r)+\frac{e\beta^2}{\gamma mR}\phi_{\mathrm{CSCF}}(r)+\frac{\gamma_{1,\mathrm{long}}v^2}{\gamma R}\ , \quad (33)$$

where we have defined the space-charge potential ϕ to vanish at (0,0), ϕ_{CSCF} is an effective centrifugal-space-charge and $\gamma_{1,\mathrm{long}}$ is the particle's energy variation from any force integrated along the direction of motion. By writing the transverse acceleration in this form, we can easily see any cancellation with the space-charge potential (if any) and we are able to maintain the same form for the centrifugal-space-charge force as we saw in Eq. (26). Note again that the first two terms on the right-hand side of Eq. (33) do not inherently lead to an emittance growth for an ultrarelativistic beam if the bend is made achromatic. If the transverse velocity is nonzero there will be a relatively small contribution from the second term. A transverse velocity can arise from the initial emittance and also from bending a beam with an energy spread. The emittance growth induced from bending a beam with transverse velocities from an initial emittance is

$$\Delta\varepsilon_{n,\beta_t}=\frac{IS\chi}{I_A\beta^2\gamma^2a^2}\varepsilon_i^2\ . \quad (34)$$

This emittance growth vanishes at high energy. The transverse velocities from a beam with an energy spread of $\Delta E/E$ in a bend of angle α will lead to an emittance growth of

$$\Delta\varepsilon_{n,\beta_t}=\frac{IS\chi}{16I_A}\left[\frac{\Delta E}{E}\alpha\right]^2\ . \quad (35)$$

Because the beam energy spread is no more than 10^{-2} (even for compression), this contribution is also typically quite small (about 0.1 μm for a 1-kA beam in a 1-rad bend that is 1 m long).

We can relate the terms in Eq. (33) to Lee's notation [8], F and G, by identifying

$$F=\frac{\phi_{\mathrm{CSCF}}(r)}{R} \quad (36)$$

and

$$G=-\frac{\phi(r)}{R}+\frac{\phi_{\mathrm{CSCF}}(r)}{R}+\frac{mc^2\gamma_{1,\mathrm{long}}}{eR}\ . \quad (37)$$

After a path length S, the transverse velocity introduced by G is (neglecting the other terms)

$$x'=\frac{dx}{dS}=\frac{\dot{x}}{\beta c}=\frac{e}{\gamma mc^2}GS\ . \quad (38)$$

The emittance growth (to be added in quadrature to the initial emittance) is then

$$\Delta\varepsilon_n=\frac{e\beta}{mc^2}S\sqrt{\langle x^2\rangle\langle G^2\rangle-\langle xG\rangle^2}\ . \quad (39)$$

Note that this equation is similar in form to Eq. (10). If there are no correlations, the emittance growth due to the transverse forces is

$$\Delta\varepsilon_n=\frac{e\beta}{mc^2}Sx_{\mathrm{rms}}G_{\mathrm{rms}} \quad (40)$$

or

$$\Delta\varepsilon_n=\frac{e\beta}{mc^2}\alpha x_{\mathrm{rms}}\langle(\phi-\phi_{\mathrm{CSCF}})^2\rangle^{1/2}\ , \quad (41)$$

where α is the bend angle. We see that this is exactly of the form of the emittance induced by an uncorrelated energy spread [Eq. (11)] if we identify the energy spread as

$$\Delta E_{\mathrm{rms}}=e\langle(\phi-\phi_{\mathrm{CSCF}})^2\rangle^{1/2}=eRG_{\mathrm{rms}}\ . \quad (42)$$

Analogous to Eq. (11), Eq. (40) can be used to find the maximum possible emittance growth for a given G_{rms}. Note also if

$$G=\frac{g_1}{R}+\frac{g_2}{R^2}+\frac{g_3}{R^3}+\cdots\ , \quad (43)$$

the emittance growth is

$$\Delta\varepsilon_n=\frac{\beta e}{mc^2}\alpha x_{\mathrm{rms}}g_{1,\mathrm{rms}}\left[1+\frac{1}{R}\frac{\langle g_1g_2\rangle}{\langle g_1^2\rangle}+\frac{1}{R^2}\left[\frac{\langle g_2^2\rangle}{2\langle g_1^2\rangle}+\frac{\langle g_1g_3\rangle}{\langle g_1^2\rangle}\right]+\cdots\right] \quad (44)$$

and only the contribution from the g_1 term does not vanish if the bend radius is made extremely large (for a fixed bend angle).

IV. BEAM UNIFORM IN AXIAL DIRECTION

In this section we will introduce a harmonic formulation which will lead to an explicit expression for the curvature term. Let us consider a ring of charge, as shown in Fig. 3. We will assume that the charge density is harmonic and that the entire ring is rotating around the origin with angular velocity $\omega=v/R$. We can sum many harmonic components to create a single short bunch of length δ in the azimuthal direction. We will also assume that we can construct the beam thickness $2a$ by superposition of various rings. This radial superposition will introduce a bunch length expansion of $2a$ after a quarter revolution, but this is not a problem because dipoles in typical achromatic bends have angles much less than one rad. For this calculation, we will assume that the ring is actually a cylinder of charge, extending uniformly in the

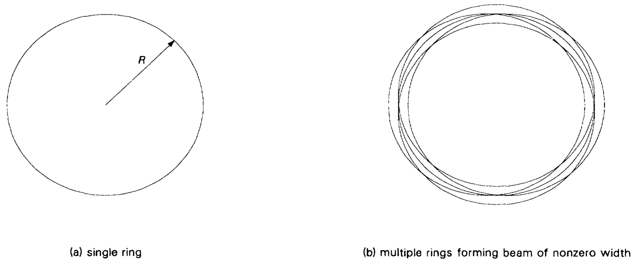

FIG. 3. (a) Single infinitesimally thin cylinder of charge (uniform in z direction) with radius R. We create an azimuthal bunch by summing up rings with various charge-density harmonics. (b) Bunch thickness is formed by superposition of slightly offset rings.

axial direction. Although this case is nonphysical, it is useful because we can explicitly solve for the electric field around the bunch. As the bend radius becomes large, we will see that the radial electric field becomes the same as if the beam was not bending. We would expect that if there is any deviation in the electric field from the usual field seen from straight-line motion that it would be demonstrated as an azimuthal modification of the radial electric field, which would be explicitly seen in this geometry. The fact that none exists will allow us to use the straight-line electric field for the more complicated geometries.

We assume that the nth current and charge density harmonics are

$$\begin{aligned} J_{\theta,n}(r,\theta) &= v\rho_n \delta(r-R)e^{in(\theta-\omega t)} , \\ \rho_n(r,\theta) &= \rho_n \delta(r-R)e^{in(\theta-\omega t)} . \end{aligned} \tag{45}$$

Since Maxwell's equations are linear and time invariant, and from symmetry we can write the field harmonic components as

$$\begin{aligned} E_{r,n}(r,\theta,z) &= \hat{E}_{r,n}(r,z)e^{in(\theta-\omega t)} , \\ E_{\theta,n}(r,\theta,z) &= \hat{E}_{\theta,n}(r,z)e^{in(\theta-\omega t)} , \\ E_{z,n}(r,\theta,z) &= \hat{E}_{z,n}(r,z)e^{in(\theta-\omega t)} , \\ B_{r,n}(r,\theta,z) &= \hat{B}_{r,n}(r,z)e^{in(\theta-\omega t)} , \\ B_{\theta,n}(r,\theta,z) &= \hat{B}_{\theta,n}(r,z)e^{in(\theta-\omega t)} , \\ B_{z,n}(r,\theta,z) &= \hat{B}_{z,n}(r,z)e^{in(\theta-\omega t)} . \end{aligned} \tag{46}$$

We use the radial component of the Maxwell equation for the curl of the magnetic field:

$$\mu \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} , \tag{47}$$

which gives

$$\hat{B}_{z,n} = -\frac{\omega r}{c^2}\hat{E}_{r,n} - i\frac{r}{n}\frac{\partial \hat{B}_{\theta,n}}{\partial z} \quad (n \neq 0) . \tag{48}$$

If we let x denote the radial location relative to the bend results, we can then write the radial Lorentz force equation for the nth harmonic as

$$F_{r,n} = e\hat{E}_{r,n}\left[\frac{1}{\gamma^2} - \beta^2\frac{x}{R}\right] - i\frac{erv}{n}\frac{\partial \hat{B}_{\theta,n}}{\partial z} \quad (n \neq 0) . \tag{49}$$

This is the harmonic version of the centrifugal space-charge force in terms of the fields. We will write this later in terms of the potentials.

Now consider the case where the beam is infinitely long and uniform in the axial direction, so we really have a rotating cylinder which is modulated in the azimuthal direction. By symmetry, the only nonzero components of the field are $E_{r,n}$ $E_{\theta,n}$, and $B_{z,n}$.

Equation (49) reduces to

$$F_{r,n} = e\hat{E}_{r,n}\left[\frac{1}{\gamma^2} - \beta^2\frac{x}{R}\right] . \tag{50}$$

By combining Eq. (47) with the Maxwell equation for the curl of the electric field, we obtain Bessel's equation for order n:

$$0 = \frac{d}{dr}\left[r\frac{d}{dr}r\hat{E}_{r,n}\right] - n^2\hat{E}_{r,n}\left[1 - \beta^2\frac{r^2}{R^2}\right] . \tag{51}$$

The general solution for the radial electric field is then

$$r\hat{E}_{r,n} = CJ_n\left[n\beta\frac{r}{R}\right] + DN_n\left[n\beta\frac{r}{R}\right] . \tag{52}$$

The problem is simplified if we assume the absence of a beam pipe. We can immediately discard the N_n solutions within the ring because the functions diverge at $r=0$. We have only two independent boundary conditions at $r=R$; the difference of the radial electric fields equals the charge density divided by the permittivity, and the tangential electric field is continuous. We know that accelerated charges radiate, thus we can assume an outward-going wave for the boundary condition at large r. The Bessel functions' asymptotic limits for large arguments are

$$\begin{aligned} J_n(z) &= \sqrt{2/\pi z}\,\cos\left[z - \frac{\pi}{4} - \frac{n\pi}{2}\right] , \\ N_n(z) &= \sqrt{2/\pi z}\,\sin\left[z - \frac{\pi}{4} - \frac{n\pi}{2}\right] . \end{aligned} \tag{53}$$

We can rewrite the fields in Eq. (52) using these boundary conditions for ultarelativistic beams as

$$r\hat{E}_{r,n} = \begin{cases} R\dfrac{\rho_n \pi n}{2\varepsilon}\left[N_n'(n)J_n\left[n\left[1+\dfrac{x}{R}\right]\right] - iJ_n'(n)J_n\left[n\left[1+\dfrac{x}{R}\right]\right]\right], & r<R \\ R\dfrac{\rho_n \pi n}{2\varepsilon}\left[J_n'(n)N_n\left[n\left[1+\dfrac{x}{R}\right]\right] - iJ_n'(n)J_n\left[n\left[1+\dfrac{x}{R}\right]\right]\right], & r>R , \end{cases} \tag{54}$$

where as before we have written $r=R+x$. Note that the large-radius electric field is given by

$$\hat{E}_{r,n}=-i\frac{R}{r}\frac{\rho_n \pi n}{2\varepsilon}J_n'(n)\sqrt{2R/\pi n r}\times e^{i(nr/R-\pi/4-n\pi/2)}e^{in(\theta-\omega t)}, \quad (55)$$

which is indeed the form for an outgoing wave. By using Eq. (53), we can find the azimuthally integrated Poynting flux (at a large radius r per unit axial length) to be

$$P_n=\frac{2R\omega}{nc^2\mu}\left[R\frac{\rho_n \pi n}{2\varepsilon}J_n'(n)\right]^2, \quad (56)$$

which, as required, is independent of r. Next, we use the harmonic charge density for a uniform slug of charge centered at $\theta=\omega t$, which is given by

$$\rho_n=\rho_r\frac{2}{n\pi}\sin\left[\frac{n\delta}{2R}\right], \quad (57)$$

where ρ_r is the bunch-current density for the ring and δ is the bunch length. Note that if the beam length is sufficiently short, ρ_n is constant for quite a few n, and the radial electric field of the infinitesimal-width bunch is dominated by the contributions from large harmonic number n. We can expand the expressions in Eq. (54) in terms of Airy functions (see the Appendix). The Airy function arguments are small for a large bend radius R, and the functions can be expanded as a cosinelike power series. The summation is then trivial, and we find that the radial electric field only exists within a cone around the bunch (where the tip of the cone lies at the bunch head). We find to first order for small bunch lengths

$$rE_r=\begin{cases}\pm\frac{R\rho_r}{2\varepsilon}, & \sqrt{\frac{2}{3}}\left[\left|\frac{x}{R}\right|\right]^{3/2}<\frac{\delta}{2R}-\left|\frac{\zeta}{R}\right| \\ \pm\frac{R\rho_r}{4\varepsilon}, & \frac{\delta}{2R}-\left|\frac{\zeta}{R}\right|<\sqrt{\frac{2}{3}}\left[\left|\frac{x}{R}\right|\right]^{3/2}<\frac{\delta}{2R}+\left|\frac{\zeta}{R}\right|, \\ 0, & \sqrt{\frac{2}{3}}\left[\left|\frac{x}{R}\right|\right]^{3/2}>\frac{\delta}{2R}+\left|\frac{\zeta}{R}\right|,\end{cases} \quad (58)$$

where the plus sign is used in the region $r<R$ and the minus sign for $r>R$, and $\zeta=R(\theta-\omega t)$. We see for the majority of the bunch the usual, nonbending contribution $\rho_r/2\varepsilon$. Unlike in purely linear motion, however, this contribution does not exist outside a cone with apex at the center of the leading edge of the bunch [note the region definitions in Eqs. (58)]. This results from the combination of causality and the lack of an inertial frame of reference. The higher-order contribution (not included) is much smaller (it goes like δ/R). It can be identified with the noncancellation of the forces from the electric and magnetic fields arising from the angular misalignment due to the retarded times. The discontinuity arises because we are assuming that we have a uniform charge density over the bunch and zero elsewhere. Note that for a large bend radius $(|x/R|)^{3/2}$ is small compared to the other terms, for all x within the bunch. Thus virtually the entire bunch will see the same radial electric fields as it would for purely linear motion, for typical bunch lengths. This is an important result. It tells us if we satisfy the top region limit in Eq. (58) that the straight-line electric field is a good approximation to use for a bunch in a bend.

Now consider the case where the bunch current varies along the length of the bunch. If the variation is sufficiently slow, we can construct the bunch out of a superposition of successively shorter, constant current bunches. As long as the current variation near the peak of the actual current profile is not too abrupt, the actual current profile can be represented by a summation of bunches not violating the region limits in Eq. (58). From superposition, the space-charge fields are the usual straight-line motion fields (plus a higher-order contribution) for arbitrary current profiles.

V. PERTURBATION ANALYSIS FOR THE SPACE-CHARGE FIELDS

In this section we will find explicit formulas for the potentials using a perturbation analysis. These potentials will then be used to derive the centrifugal-space-charge force as a function of the bend radius and beam parameters in the next section. We will do this by assuming that the azimuthal component of the vector potential is very close to β/c times the scalar potential and do a perturbation analysis based on the wave equations for the scalar and vector potentials. In particular, we will find an expression for the potentials for a ring of current. We can then use this expression as a Green's function for a uniform-density beam with circular cross section.

As before, we assume a ring of current at a radius R with a time dependence of $e^{in(\theta-\omega t)}$. Let us now also assume that the vector potential for harmonic n is given by

$$\mathbf{A}_n=(\delta A_{r,n},(\beta/c)\phi_n+\delta A_{\theta,n},0) \quad (59)$$

(A_z must remain zero because we have no current flow in the z direction). The wave equations for the scalar and vector potentials away from the ring of current,

$$\nabla^2\phi_n e^{in\theta}+\frac{n^2\omega^2}{c^2}\phi_n e^{in\theta}=0 \tag{60}$$

and

$$\nabla^2 \mathbf{A}_n e^{in\theta}+\frac{n^2\omega^2}{c^2}\mathbf{A}_n e^{in\theta}=\mathbf{0} , \tag{61}$$

lead to these expressions:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi_n}{\partial r}-n^2\phi_n+\frac{\partial^2\phi_n}{\partial z^2}+\frac{n^2\omega^2}{c^2}\phi_n=0 , \tag{62}$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\left[\frac{\beta}{c}\phi_n+\delta A_{\theta,n}\right]-n^2\left[\frac{\beta}{c}\phi_n+\delta A_{\theta,n}\right]+\frac{\partial^2}{\partial z^2}\left[\frac{\beta}{c}\phi_n+\delta A_{\theta,n}\right]+\frac{n^2\omega^2}{c^2}\left[\frac{\beta}{c}\phi_n+\delta A_{\theta,n}\right]+\frac{2in}{r^2}\delta A_{r,n}-\frac{\frac{\beta}{c}\phi_n+\delta A_{\theta,n}}{r^2}=0 , \tag{63}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\delta A_{r,n}-n^2\delta A_{r,n}+\frac{\partial^2}{\partial z^2}\delta A_{r,n}+\frac{n^2\omega^2}{c^2}\delta A_{r,n}-\frac{2in}{r^2}\left[\frac{\beta}{c}\phi_n+\delta A_{\theta,n}\right]+\frac{\delta A_{r,n}}{r^2}=0 . \tag{64}$$

These expressions reduce to

$$\frac{1}{r}\frac{\partial}{\partial r}\delta A_{r,n}+\left[\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right]\delta A_{r,n}+\frac{n^2 2x}{rR^2}\delta A_{r,n}-\frac{2in}{r^2}\left[\frac{\beta}{c}\phi_n+\delta A_{\theta,n}\right]-\frac{\delta A_{r,n}}{r^2}=0 \tag{65}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\delta A_{\theta,n}+\left[\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right]\delta A_{\theta,n}+\frac{n^2 2x}{rR^2}\delta A_{\theta,n}+\frac{2in}{r^2}\delta A_{r,n}-\frac{\left[\frac{\beta}{c}\phi_n+\delta A_{\theta,n}\right]}{r^2}=0 . \tag{66}$$

For sufficiently low harmonic number n,

$$\left[\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right]\delta A_{r,n}=\frac{2in}{r^2}\frac{\beta}{c}\phi_n , \tag{67}$$

$$\left[\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial z^2}\right]\delta A_{\theta,n}=\frac{1}{r^2}\frac{\beta}{c}\phi_n , \tag{68}$$

and for very large n,

$$\delta A_{r,n}=\frac{iR}{nx}\frac{\beta}{c}\phi_n , \tag{69}$$

$$\delta A_{\theta,n}=\frac{R}{n^2x}\frac{\beta}{c}\phi_n . \tag{70}$$

The other important relation is the Lorentz gauge equation:

$$\nabla\cdot(\mathbf{A}e^{in(\theta-\omega t)})=-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi e^{in(\theta-\omega t)} , \tag{71}$$

which leads to this additional expression:

$$\frac{\beta}{c}\phi_n\frac{x}{R}+\frac{i}{n}r\frac{\partial\delta A_{r,n}}{\partial r}+\frac{i}{n}\delta A_{r,n}-\delta A_{\theta,n}=0 . \tag{72}$$

From the results in the preceding section, we can assume that the scalar potential is very close to the scalar potential of the straight-line beam,

$$\phi_n=\frac{I_n}{2\pi\varepsilon\beta c}\ln\frac{\rho}{\rho_0} , \tag{73}$$

where I_n is the harmonic current, $\rho^2=x^2+z^2$, and ρ_0 is the radius of the beam pipe. After keeping just the lowest-order terms, the Lorentz gauge expression can be integrated to yield in the low n regime

$$\delta A_{\theta,n}=\frac{in}{4Rr}\frac{I_n}{2\pi\varepsilon c^2}\left\{\rho^2\ln\frac{\rho^2}{\rho_0^2}-\rho^2+f(z)\right\} , \tag{74}$$

where the function $f(z)$ is unspecified. This function can in turn be found by taking the transverse derivatives in the wave equations, which leads to

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right]\delta A_{r,n}=\frac{in}{4Rr}\frac{I_n}{2\pi\varepsilon c^2}\left\{4\ln\frac{\rho^2}{\rho_0^2}+4\frac{\rho^2}{\rho_0^2}+f''(z)\right\} , \tag{75}$$

or, from using Eq. (67),

$$\delta A_{r,n}=\frac{in}{4Rr}\frac{I_n}{2\pi\varepsilon c^2}\left\{\rho^2\ln\frac{\rho^2}{\rho_0^2}-\rho^2-2z^2\right\}. \tag{76}$$

In this limit (small n), we also see

$$\delta A_{\theta,n}=-\frac{1}{8Rr}\frac{I_n}{2\pi\varepsilon c^2}\left\{\rho^2\ln\frac{\rho^2}{\rho_0^2}-\rho^2-2z^2\right\}. \tag{77}$$

Now we have completely specified the fields and can calculate the emittance growth due to different effects. Note that the expression for $\delta A_{\theta,n}$ only depends on the harmonic number through I_n. For this component, the harmonic summation is trivial, and gives

$$\delta A_{\theta}=-\frac{1}{8Rr}\frac{I}{2\pi\varepsilon c^2}\left\{\rho^2\ln\frac{\rho^2}{\rho_0^2}-\rho^2-2z^2\right\}. \tag{78}$$

Equation (78) must hold for the dc case also; thus this perturbation (with I now the dc current) is also the correction to the vector potential we assumed in the dc analysis [$\mathbf{A}=(0,(\beta/c)\phi,0)$]. This perturbation leads to an error in the equation of motion that depends on $1/R^2$.

The low n regime is defined by when

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right]\delta A_{r,n}>\frac{2n^2x\beta^2}{Rr^2}\delta A_{r,n} \tag{79}$$

[check the wave equation, Eq. (65)], or when

$$n<\frac{2R}{\beta}\left[\frac{r}{x}\right]^{1/2}\left[\frac{\ln(\rho/\rho_0)}{2\rho^2\ln(\rho/\rho_0)-\rho^2-2z^2}\right]^{1/2}. \tag{80}$$

Equations (73), (76), and (77) specify the lowest-order potential components. Of course, the next-order corrections in the scalar potential might very well be larger than either $\delta A_{r,n}$ or $\delta A_{\theta,n}$; however, we will see in the next section that an error in the scalar potential will not contribute to the transverse equation of motion of the particles.

From the preceding section, we know that the radial electric field must be that for straight-line motion to lowest order. In order to check the validity of using Eq. (73) for the scalar potential we can explicitly calculate the radial electric field from the potentials [Eqs. (73) and (76)] and compare it to the radial field for straight-line motion. Using

$$E_{r,n}=-\frac{\partial}{\partial r}\phi_n+in\omega\delta A_{r,n} \tag{81}$$

we find

$$E_{r,n}=-\frac{I_n}{2\pi\varepsilon\beta c}\left[\frac{1}{x}+\frac{n^2}{R^2r}\left\{\rho^2\ln\frac{\rho^2}{\rho_0^2}-\rho^2-z^2\right\}\right]. \tag{82}$$

The modification to the straight-line radial electric field is always small [check Eq. (80)] and thus Eq. (73) is a fine approximation to use for the scalar potential.

VI. LORENTZ FORCE EQUATION AND THE TRANSVERSE EQUATION OF MOTION

In this section we will calculate the transverse motion of the beam required for estimating the emittance growth of an electron bunch traveling in a circular path as it would within a bending magnet. We will also show that these fields are consistent with Piwinski's results. We will start with the Lorentz force equation in terms of the vector and scalar potentials and the transverse equation of motion in the beam's local frame of reference.

The Lorentz force equation for the nth harmonic can be written in terms of the vector and scalar potentials (by direct expansion of the field components in the potentials) as

$$\begin{aligned}F_{r,n}&=E_{r,n}+vB_{z,n}\\&=e\left[-\frac{1}{\gamma^2}\frac{\partial\phi_n}{\partial r}+\beta^2\frac{\phi_n}{r}+v\frac{\partial\delta A_{\theta,n}}{\partial r}\right.\\&\qquad\left.+\frac{v}{r}\delta A_{\theta,n}+\frac{invx}{rR}\delta A_{r,n}\right].\end{aligned} \tag{83}$$

The first term on the right-hand side of Eq. (83) is the usual space-charge term that appears in straight-line motion. The second term is the same as we saw in the earlier derivation for the dc case, and will again cancel another term in the dynamical equation. The third and fourth terms lead to terms (after the summation over harmonic number) that are on the order of $1/R^2$ (third term) and $1/R^3$ (fourth term) and we will neglect these terms. The fifth term will lead to a more interesting result (after the required harmonic summation) and we will follow this term in more detail.

The transverse equation of motion [using Lee's notation and Eq. (33)] is

$$\ddot{x}=\frac{v^2}{R}\left[-\frac{x}{R}+\frac{x^2}{2R^2}-\frac{x^3}{6R^3}+\cdots\right]+\frac{e}{m\gamma}\frac{d\phi}{dr}\left[\beta_t^2-\frac{1}{\gamma^2}\right]+\frac{e\beta^2}{m\gamma}G, \tag{84}$$

where

$$G=\gamma_1\frac{mc^2}{R}+\frac{F_{sc}}{e\beta^2} \tag{85}$$

and the space-charge force F_{sc} consists of the last four terms in the Lorentz force, summed over all harmonics.

The first term in Eq. (84) is not necessarily small, but does not lead to an emittance growth if the bend is achromatic. The second term on the right-hand side also appears for straight-line motion and is small, and we will ignore it. The G term is more complicated. If a particle's energy is not the nominal energy, but does not change, there will not be an emittance growth if the bend is achromatic. However, a change in a particle's energy can occur from work done on the particle in both the transverse and azimuthal directions. Assuming that the bunch does not deform appreciably along the bend, we can write the change in a particle's energy within a bend as

$$\gamma_1 = \frac{eE_\theta}{mc^2}S + \frac{e}{mc^2}\int E_r dl \ , \tag{86}$$

where the S is the distance along the bend that the bunch has gone and the integral is from the center of the bunch to the particle's position. In this case, the integral can be rewritten for each harmonic as

$$\int E_{r,n} dl = -\phi_n(x,z) - in\omega \int_{(0,0)}^{(x,z)} \delta A_{r,n} dl \ . \tag{87}$$

After dropping the achromatic terms, we can rewrite the dynamical equation (doing the harmonic summation where trivial) as

$$\begin{aligned}\gamma\ddot{x} = {} & \frac{\beta^2 eE_\theta S}{mR} - \frac{\beta^2 e\phi(x,z,R\theta)}{mR} \\ & - \sum_{n=1}^{n_{\max}} \frac{\beta^2 einv \int_{(0,0)}^{(x,z)} \delta A_{r,n} dl}{mR^2} \\ & - \frac{e}{m\gamma^2}\frac{d\phi}{dr} + \beta^2 \frac{e\phi(x,z,R\theta)}{mr} \\ & + \sum_{n=1}^{n_{\max}} e\frac{invx}{mrR}\delta A_{r,n} \ .\end{aligned} \tag{88}$$

There is no significant contribution to the centrifugal-space-charge force when $n > n_{\max}$. As in the dc case, the second and fifth terms cancel exactly, and we are left with the usual energy dependent term plus three noncanceling energy-independent terms:

$$\begin{aligned}\gamma\ddot{x} = {} & -\frac{e}{m\gamma^2}\frac{d\phi}{dr} + \frac{\beta^2 eE_\theta S}{mR} \\ & + \sum_{n=1}^{n_{\max}} \frac{einv}{mR}\left[\frac{x}{r}\delta A_{r,n} - \frac{\beta^2}{R}\int_{(0,0)}^{(x,z)} \delta A_{r,n} dl\right] \ .\end{aligned} \tag{89}$$

Because these terms will lead to fundamentally different effects, we will solve for the emittance growths separately. Before that, though, let us look at Piwinski's results.

Comparison to Piwinski's results

Piwinski showed that the electric field from the nth harmonic of the current distribution in a beam pipe of height h with inner radius r_1 and outer radius r_2 can be written as (for $r < R$; the case where $r > R$ is similar)

$$\begin{aligned}\mathbf{E} = {} & A_1(\mathbf{e}_1 K_n'(ar_1) - \mathbf{e}_2 I_n'(ar_1)) \\ & + A_2(\mathbf{e}_3 K_n(ar_1) - \mathbf{e}_4 I_n(ar_1))\end{aligned} \tag{90}$$

where I_n and K_n are the modified Bessel functions,

$$\begin{aligned}\mathbf{e}_1 &= \left[-\frac{in}{ar}I_n(ar)\cos kz, I_n'(ar)\cos kz, 0\right]e^{in(\theta-\omega t)} \ , \\ \mathbf{e}_2 &= \left[-\frac{in}{ar}K_n(ar)\cos kz, K_n'(ar)\cos kz, 0\right]e^{in(\theta-\omega t)} \ , \\ \mathbf{e}_3 &= \left[kI_n'(ar)\cos kz, \frac{ink}{ar}I_n(ar)\cos kz, -aI_n(ar)\sin kz\right]e^{in(\theta-\omega t)} \ , \\ \mathbf{e}_4 &= \left[kK_n'(ar)\cos kz, \frac{ink}{ar}K_n(ar)\cos kz, -aK_n(ar)\sin kz\right]e^{in(\theta-\omega t)} \ ,\end{aligned} \tag{91}$$

k is the vertical wave number ($k = \pi m/h$, where m is an odd integer),

$$a^2 = k^2 - \frac{n^2\beta^2}{R^2} \ , \tag{92}$$

and the coefficients A_1 and A_2 are given in terms of the harmonic charge distribution λ_{nk} by

$$\begin{aligned}A_1 &= in\lambda_{nk} v\beta Z_0 \frac{I_n'(ar_2)K_n'(aR) - K_n'(ar_2)I_n'(aR)}{I_n'(ar_1)K_n'(ar_2) - K_n'(ar_1)I_n'(ar_2)} \ , \\ A_2 &= \lambda_{nk} cZ_0 \frac{kR}{a}\frac{I_n'(ar_2)K_n(aR) - K_n(ar_2)I_n(aR)}{I_n(ar_1)K_n(ar_2) - K_n(ar_1)I_n(ar_2)} \ .\end{aligned} \tag{93}$$

Using these expressions and the curl equation for the electric field, we can use Eq. (49) to find the radial force:

$$\frac{F_{r,nk}}{e}=\frac{k\cos kz}{\gamma^2}A_1(K_n(ar)I_n'(ar_1)-I_n(ar)K_n'(ar_1))\frac{in}{kar}+\frac{k\cos kz}{\gamma^2}A_2(K_n(ar_1)I_n'(ar)-I_n(ar_1)K_n'(ar))$$
$$+k\cos(kz)A_1(K_n(ar)I_n'(ar_1)-I_n(ar)K_n'(ar_1))\frac{in}{kar}\left[\beta^2\frac{x}{R}+\frac{k^2Rr}{n^2}\right]$$
$$+k\cos(kz)\beta^2A_2(K_n(ar_1)I_n'(ar)-I_n(ar_1)K_n'(ar))\ . \tag{94}$$

Although this equation looks complicated, it simplifies if we consider the limits where the bunch length is either long or short compared to h. In either limit, it is easy to show that

$$A_1\approx A_2\frac{h^2}{\delta^2}k\ . \tag{95}$$

Thus, if the beam length is long compared to the beam pipe height ($h/\delta\ll 1$ and $n/kR\ll 1$), only the A_2 terms and the A_1 term involving k/n are important. Conversely, if the beam length is short, only the terms with A_1 contribute. In the long bunch limit, Eq. (94) reduces to the form of Eq. (26)—the centrifugal-space-charge potential is just the space-charge potential

$$\phi_{\rm CSCF}=\phi=k\cos(kz)\lambda_{nk}Z_0\frac{kR}{a}c$$
$$\times\left[\frac{I_n'(ar_2)K_n(aR)-K_n(ar_2)I_n(aR)}{I_n(ar_1)K_n(ar_2)-K_n(ar_1)I_n(ar_2)}[I_n(ar_1)K_n'(ar)-K_n(ar_1)I_n'(ar)]\right.$$
$$\left.-\frac{I_n'(ar_2)K_n'(aR)-K_n'(ar_2)I_n'(aR)}{I_n'(ar_1)K_n'(ar_2)-K_n'(ar_1)I_n'(ar_2)}[I_n(ar)K_n'(ar_1)-K_n(ar)I_n'(ar_1)]\right]\ . \tag{96}$$

This is the limit Piwinski examined (he assumed ar is very large and real) and in this limit the dc case is recovered (as one would expect).

In the short bunch limit (keeping just the A_1 terms), Eq. (94) no longer reduces to the form of Eq. (26). Thus, as the bunch length becomes smaller than the beam pipe height, the cancellation of the centrifugal-space-charge force and the effect of the variation of the beam kinetic energy vanish and an emittance growth will result. This is an important result, and explains how the noncancellation arises for short bunch lengths.

VII. EMITTANCE ESTIMATES

In this section we will estimate the emittance of a bunch traveling in a bend. Just as in the first section, we will calculate two contributions. The first contribution will be from the net transverse space-charge forces and the second will be from the effect of the electric field in the direction of motion of the bunch. The first effect will lead to a relatively small, but not necessarily negligible emittance growth, that scales as the bend angle times the beam radius to the $\frac{5}{2}$ power divided by the beam length and the square root of the bend radius. Additionally, if the beam is not being bunched and if the betatron motion is symmetric (or negligible), an achromatic bend will prevent any emittance growth from this effect. On the other hand, there will always be an emittance growth from the second effect. The emittance growth from this other effect will scale as the bend angle squared times the beam radius squared divided by the bunch length. It can be relatively large for short bunches in large-angle achromatic bends. In both cases, the emittance growth can be made negligible by sufficiently strong focusing of the beam through the bend. On the other hand, in both cases the emittance growth scales as the square of the peak current for a constant-charge bunch. This will provide a peak current limit for bunch compression for a given beam radius.

A. Harmonic summation of the $\delta A_{r,n}$ terms

First let us sum the radial vector potential terms over all harmonics. We have to be somewhat careful, because the vector potential solution, Eqs. (73), (76), and (77), is only valid for a line of charge. We will first do the harmonic summation and then the integration over the beam's cross section. Using Eq. (33) we can define an effective centrifugal-space-charge potential $\phi_{\rm CSCF}$ and a net centrifugal-space-charge potential $R\hat{G}$ by

$$\hat{G}=\frac{\phi_{\rm CSCF}}{R}-\frac{\phi}{R}$$
$$=-\sum_{n=1}^{n_{\max}}\frac{inv\int_{(0,0)}^{(x,z)}\delta A_{r,n}dl}{R^2}+\sum_{n=1}^{n_{\max}}\frac{invx}{\beta^2rR}\delta A_{r,n}\ . \tag{97}$$

We drop the energy-dependent force and the other terms in the equation of motion that do not lead to an emittance growth. The transverse equation of motion becomes

$$\gamma\ddot{x}=e\beta^2\frac{\hat{G}}{m}+\frac{\beta^2 eE_\theta S}{mR} . \qquad (98)$$

If the line bunch of length δ has uniform current I_l, the current harmonics are

$$I_n=I_l\frac{2}{n\pi}\sin\frac{n\delta}{2R} . \qquad (99)$$

Since β is very close to unity, $\hat{G}$ can be rewritten as

$$\hat{G}=-\sum_{n=1}^{n_{\max}}\frac{n}{r^2R^2}\frac{I_l}{4\pi^2\varepsilon c}\sin\left[\frac{n\delta}{2R}\right]f(x,z)e^{in(\theta-\omega t)} , \qquad (100)$$

where we have lumped the various transverse functions in f:

$$f(x,z)=x\rho^2\ln\frac{\rho^2}{\rho_0^2}-x\rho^2-2xz^2 -\frac{1}{3}\left\{\rho^3\ln\frac{\rho^2}{\rho_0^2}-\frac{5}{3}\rho^3-2z^3\right\} . \qquad (101)$$

We will estimate the emittance growth from this effect by (1) doing the harmonic summation, (2) making an estimate for the function $\hat{G}$ after examining typical numerical summations over cylindrical, uniform beams, (3) assuming a parabolic current profile along the bunch, and (4) using Eqs. (39) and (40) to relate $\hat{G}_{\rm rms}$ to the emittance growth.

The summation over harmonics can be approximated well by an integral over n, which leads to

$$\hat{G}=-\frac{1}{r^2R^2}\frac{I_l}{8\pi^2\varepsilon\beta c}f(x,z)\int_1^{n_{\max}}\left\{n\sin\left[n\left[\frac{\delta}{2R}+\frac{\zeta}{R}\right]\right]+n\sin\left[n\left[\frac{\delta}{2R}-\frac{\zeta}{R}\right]\right]\right\}dn$$

$$=\frac{1}{r^2R^2}\frac{I_l}{8\pi^2\varepsilon\beta c}f(x,z)\left[\frac{n_{\max}}{\frac{\delta}{2R}+\frac{\zeta}{R}}\cos\left[n_{\max}\left[\frac{\delta}{2R}+\frac{\zeta}{R}\right]\right]+\frac{n_{\max}}{\left[\frac{\delta}{2R}-\frac{\zeta}{R}\right]}\cos\left[n_{\max}\left[\frac{\delta}{2R}-\frac{\zeta}{R}\right]\right]\right] , \qquad (102)$$

where as before we have used $\zeta=R(\theta-\omega t)$. Within the bunch itself, $|\zeta|<\delta/2$, and $\hat{G}$ is close to

$$\hat{G}=\frac{1}{R\sqrt{rx}\,\delta}\frac{I_l}{2\pi^2\varepsilon c}f(x,z)\times\left[\frac{\ln(\rho^2/\rho_0^2)}{\rho^2\ln(\rho^2/\rho_0^2)-\rho^2-2z^2}\right]^{1/2} . \qquad (103)$$

Recall that this $\hat{G}$ is for a line source charge. We can numerically integrate $\hat{G}$ over a full uniform, circular cross section (see Figs. 4–7) and we find that the effective potential can be approximated by

$$\hat{G}_{\rm integrated}=2.16\frac{a^{3/2}}{R\sqrt{R}\,\delta}\frac{x}{a}\frac{I}{4\pi\varepsilon c}\ln\left[\frac{\rho_0}{a}\right] . \qquad (104)$$

As before, we can add several uniform current bunches together to generate arbitrary current profiles. From the linearity of the fields this is equivalent to letting the current I in Eq. (104) be a function of position within the bunch. We will assume the bunch has a parabolic current profile with peak current I_p:

$$I(\zeta)=I_p\left[1-\left[\frac{2\zeta}{\delta}\right]^2\right],\quad |\zeta|<\frac{\delta}{2} . \qquad (105)$$

In this case, the rms value of $\hat{G}$ becomes

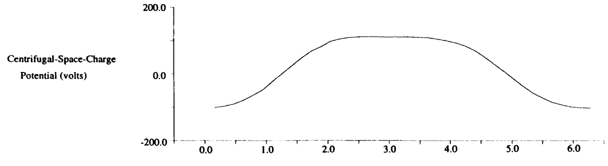

FIG. 4. Normalized integrated centrifugal space-charge potential

$$\frac{I_l}{2\pi^2\varepsilon c a^{3/2}x^{1/2}}f(x,z)\left[\frac{\ln(\rho^2/\rho_0^2)}{\rho^2\ln(\rho^2/\rho_0^2)-\rho^2-2z^2}\right]^{1/2}$$

at a radius a versus observation angle φ for a 1-A beam of radius a. Ratio of beam-pipe radius to beam radius is 10.

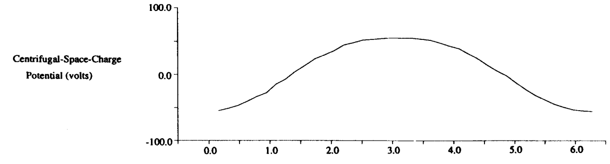

FIG. 5. Normalized centrifugal space-charge potential

$$\frac{I_l}{2\pi^2\varepsilon c a^{3/2}x^{1/2}}f(x,z)\left[\frac{\ln(\rho^2/\rho_0^2)}{\rho^2\ln(\rho^2/\rho_0^2)-\rho^2-2z^2}\right]^{1/2}$$

at a radius $a/2$ versus observation angle φ for a 1-A beam of radius a. Ratio of beam-pipe radius to beam radius is 10.

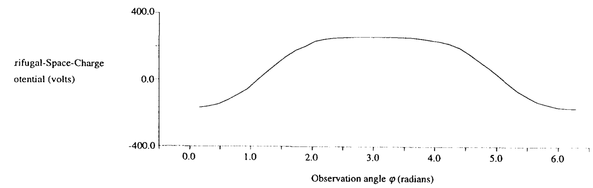

FIG. 6. Normalized integrated centrifugal space-charge potential

$$\frac{I_l}{2\pi^2 \varepsilon c a^{3/2} x^{1/2}} f(x,z) \left[\frac{\ln(\rho^2/\rho_0^2)}{\rho^2 \ln(\rho^2/\rho_0^2) - \rho^2 - 2z^2} \right]^{1/2}$$

at a radius a versus observation angle φ for a 1-A beam of radius a. Ratio of beam-pipe radius to beam radius is 100.

$$G_{\text{rms}} = 1.08 \frac{a^{3/2}}{R\sqrt{R}\,\delta} \frac{I_p}{4\pi\varepsilon c} \ln\left[\frac{\rho_0}{a}\right] . \tag{106}$$

From Eq. (39), this leads to an emittance growth in the bend plane (to be added in quadrature to the initial emittance) of

$$\Delta\varepsilon_n = 0.14 \frac{a^{3/2}}{\sqrt{R}\,\delta} \ln\left[\frac{\rho_0}{a}\right] \frac{I_p}{I_A} a\alpha . \tag{107}$$

The emittance growth in the direction normal to the bend plane is higher order and negligible [the energy-independent force is comparable to the third term in the Lorentz force equation, Eq. (83), which we subsequently dropped]. The emittance growth indicated in Eq. (107) is similar to that for the "classical" case [where the space-charge force is energy dependent, Eq. (19)], but with an additional leading factor involving the beam radius, bend radius, and bunch length and one involving the ratio of the beam-pipe radius to the beam radius and missing any depending on the beam energy. Please note the region definitions in the preceding section where we explicitly calculated the fields for the rotating cylinder of charge. That result implies that if this leading factor is unity or greater, the space-charge fields around the bunch are higher order in $1/R$ and small, and the approximation we used for the scalar potential (being the same as for straight-line motion) fails. Thus. Eq. (107) is only valid if the leading term is less than unity, and there is no significant emittance growth if that term is unity or greater. Note that the emittance growth scales roughly as the $\frac{5}{2}$ power of the beam radius and that for a constant charge, the emittance growth increases as the square of the peak current (since δ also decreases).

Typical numbers for these parameters for short-wavelength FELs and linear colliders are as follows: beam radius of 1 mm, bend radius of 1 m, and bunch length of 1 ps (0.3 mm). The leading factor for this case is about 0.1. It can be made somewhat smaller by increasing the bend radius, but not significantly because the bend radius comes in as the square root. For a bunch current of 1 kA in a beam pipe radius of 1 cm, this emittance growth is about 3 μm per mm of beam radius per radian of bend angle. On the other hand, if the beam radius is reduced to 100 μm, the emittance growth would drop to less than 0.03 μm per kA of peak current. It should be noted that this emittance growth has been reduced by the near cancellation of the $\langle x^2\rangle\langle G^2\rangle$ and the $\langle xG\rangle^2$ terms in Eq. (39). This cancellation primarily occurs because we assumed a uniform radial charge density which leads to G being linear with x; this cancellation would not occur for all possible radial charge distributions, and a more conservative estimate for the emittance growth would be made using Eq. (40) and G_{rms}. For this case, the maximum possible emittance growth would be about three times higher than indicated in Eq. (107).

As a last calculation for the net centrifugal-space-charge force, we will find the emittance generated while compressing a bunch of initial current I_0 to a final current I_f. Typical compression schemes use multiple dipoles [2], leading to nonzero dispersion. We assume that there are enough dipoles that the bunch current within the compressor can be expressed as

$$I = \frac{I_0}{1 - \dfrac{\alpha}{\alpha_0}} , \tag{108}$$

where the normalized bend angle is given in terms of the total bend angle α_f by

$$\alpha_0 = \frac{\alpha_f}{1 - \dfrac{I_0}{I_f}} . \tag{109}$$

The emittance growth as the bunch is compressed can be found by integrating Eq. (107). The emittance growth from a compression stage is given by

$$\Delta\varepsilon_{n,\text{bunching}} = 0.14 \frac{a^{3/2}}{\sqrt{R}\,\delta_0} \alpha_0 a \ln\left[\frac{\rho_0}{a}\right] \frac{I_f}{I_A} , \tag{110}$$

where the final current is the current after compression and δ_0 is the bunch length before compression. The emittance growth is larger, though comparable, if the transverse distribution is not uniform. As an example, bunching a 1-mm beam from 250 A to 1 kA in a 1-rad bend

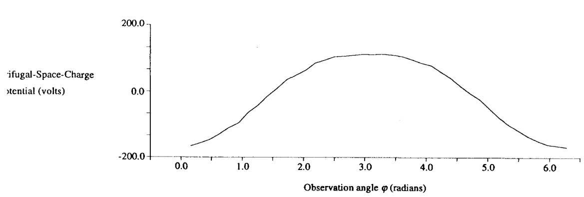

FIG. 7. Normalized integrated centrifugal space-charge potential

$$\frac{I_l}{2\pi^2 \varepsilon c a^{3/2} x^{1/2}} f(x,z) \left[\frac{\ln(\rho^2/\rho_0^2)}{\rho^2 \ln(\rho^2/\rho_0^2) - \rho^2 - 2z^2} \right]^{1/2}$$

at a radius $a/2$ versus observation angle φ for a 1-A beam of radius a. Ratio of beam-pipe radius to beam radius is 100.

with a beam-pipe radius to beam radius ratio of 10 creates about 3.4 μm of emittance growth. Again note the strong dependence of the emittance growth on the beam radius and that this assumes a uniform radial charge density.

B. Energy gain in the direction of motion

The second emittance growth mechanism results from the work done on the bunch by the electric field in the azimuthal direction, the direction of motion. As noted before, the energy change of a particle at $(x,z,R\theta)$ within the bunch due to this field after a path length S is given by

$$\Delta\gamma mc^2 = eE_\theta S \ . \qquad (111)$$

The total energy change, $\gamma_1 mc^2$, is this term plus the contribution from the transverse electric field. In order to calculate the azimuthal field we start with the components from the individual harmonics:

$$E_{\theta,n} = -\frac{1}{r}\frac{\partial\phi_n}{\partial\theta} - \frac{\partial A_{\theta,n}}{\partial t} \ . \qquad (112)$$

After substituting in the values for the vector potential we find

$$E_{\theta,n} = -\frac{in\phi_n}{r\gamma^2} + \beta^2\frac{in\phi_n}{rR}x + in\omega\delta A_{\theta,n} \ . \qquad (113)$$

The first term is the same as we find for linear motion (in/r would be replaced by $in\beta$ for linear motion with an $e^{in(\beta z-\omega t)}$ dependence). It will vanish for a sufficiently relativistic beam. The third term is of order $1/R$ higher than the second term (recall that $\omega = v/R$) and can be neglected. Initially, the harmonic summation looks complicated but since we know that for the straight-line motion the first term sums up to the usual longitudinal electric field we can write the energy-independent second term (for a line of current I) as

$$E_{\theta,l} = \frac{dI}{d\zeta}\frac{1}{2\pi\varepsilon\beta c}\frac{x}{R}\ln\frac{\rho}{\rho_0} \ , \qquad (114)$$

which leads to

$$E_{\theta,l} = \frac{4I_l\zeta}{\pi\varepsilon\beta c\delta^2}\frac{x}{R}\ln\frac{\rho}{\rho_0} \ , \qquad (115)$$

for the parabolic current profile [Eq. (105)]. As before we have to integrate over all source positions to find the overall azimuthal field

$$E_\theta = \frac{4I_p(\zeta)\zeta}{\pi^2\varepsilon\beta c\delta^2 a^2}\int_0^a\int_0^{2\pi}\frac{x-\rho\cos\varphi}{R}\ln\frac{\sqrt{(x-\rho\cos\varphi)^2+(z-\rho\sin\varphi)^2}}{\rho_0}\rho\, d\rho\, d\varphi \ . \qquad (116)$$

This expression can also be conveniently numerically integrated for a beam with uniform, circular cross section (Figs. 8–11), and leads to

$$E_\theta = 15.2\frac{I_p\zeta}{4\pi c\varepsilon}\ln\left[\frac{\rho_0}{a}\right]\frac{x}{a}\frac{a}{R\delta^2} \ . \qquad (117)$$

Because the longitudinal force is linear with ζ, $\langle x\Delta E\rangle$ is zero and the emittance growth is given by Eq. (11). The rms energy spread caused by this field is

$$\Delta E_{\rm rms} = 1.52\frac{eI_p}{4\pi c\varepsilon}\ln\left[\frac{\rho_0}{a}\right]\frac{a}{R\delta}S \ . \qquad (118)$$

The emittance growth within the bend for an angle $d\alpha$ can be expressed as

$$d\varepsilon_n = 0.76\frac{I_p}{I_A}a\ln\left[\frac{\rho_0}{a}\right]\frac{a}{\delta}\alpha\, d\alpha \qquad (119)$$

and, after integrating along the beam path within the bend, we find that the emittance growth in the bend plane (to be added in quadrature to the initial emittance) is

$$\Delta\varepsilon_n = 0.38\alpha^2\frac{I_p}{I_A}\ln\left[\frac{\rho_0}{a}\right]a\frac{a}{\delta} \ . \qquad (120)$$

This emittance growth is per achromatic section of the

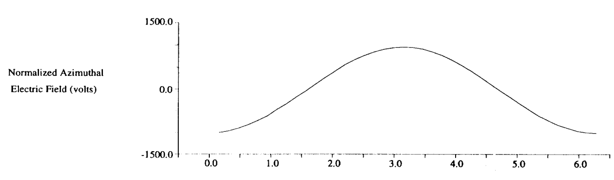

FIG. 8. Normalized integrated azimuthal electric field $(4I_l/\pi\varepsilon\beta c)(x/a)\ln(\rho/\rho_0)$ at a radius a versus observation angle φ for a 1-A beam of radius a. Ratio of beam-pipe radius to beam radius is 10.

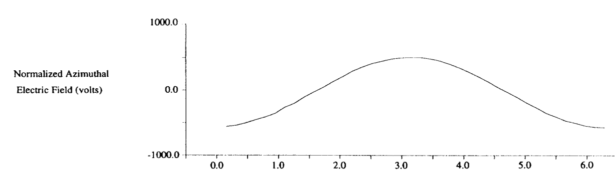

FIG. 9. Normalized integrated azimuthal electric field $(4I_l/\pi\varepsilon\beta c)(x/a)\ln(\rho/\rho_0)$ at a radius $a/2$ versus observation angle φ for a 1-A beam of radius a. Ratio of beam-pipe radius to beam radius is 10.

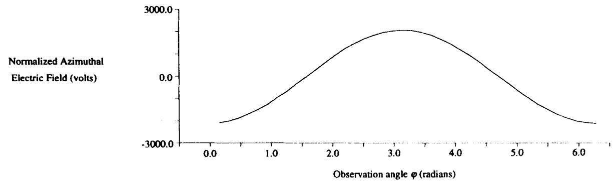

FIG. 10. Normalized integrated azimuthal electric field $(4I_l/\pi\varepsilon\beta c)(x/a)\ln(\rho/\rho_0)$ at a radius a versus observation angle φ for a 1-A beam of radius a. Ratio of beam-pipe radius to beam radius is 100.

bend. For example, if there are four separate 20° achromatic bends making up an 80° bend, 20° should be used in Eq. (120) for the angle, and the result multiplied by 4 to find the total emittance growth.

This emittance growth can be quite large if the bend angle and the beam size are not small. For example, for a 1-ps-long, 1-mm-radius, 1-kA bunch in a 1-rad bend with a 1-cm beam pipe there is 200 μm times the bend angle squared of emittance growth. For a 180° bend, there would be nearly 2000 μm of emittance growth if the beam-pipe radius is ten times the beam radius. Note that the emittance growth scales as the beam radius squared and, as before, as the square of the peak current for a constant-charge bunch. In principle, the emittance growth can be made negligible if the beam is tightly focused.

VIII. CONCLUSION

We have analyzed the space-charge fields in a bend and found that they lead to energy-independent transverse and azimuthal forces and thus an energy independent emittance growth in the plane of the bend. We also found that the space-charge fields are very similar to the ones for straight-line motion as long as the bunch length is sufficiently short, defined by the region limits in Eq. (58). An additional bonus to this calculation is that we also derived the correction terms to the vector potential in terms of the scalar potential.

The practical consequence of this effect is that magnetic bunch compressors for future short-wavelength FELs and linear colliders need to be carefully designed even if the compression is done at extremely high energy. The dominant emittance growth mechanism scales as the square of the beam radius, the square of the bend angle, and the square of the peak current (for a constant-charge bunch). Thus the growth can be made small by focusing the beam sufficiently tightly and by breaking the total bend up into many separate, small-angle achromatic bends. In a chicane magnetic compressor the difference in path length for particles with different energies scales as the path length through the compressor times the bend angle squared. Thus by increasing the bend radius and decreasing the bend angle these effects could be made negligible while preserving the bunch compression. The effects discussed in this paper, however, will lead to a practical limit for the maximum peak current attainable at a given energy from bunch compression.

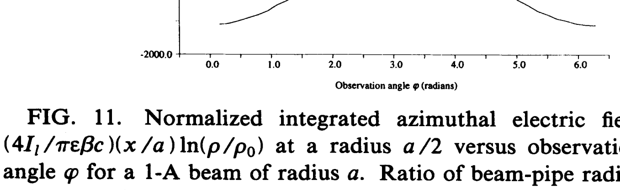

FIG. 11. Normalized integrated azimuthal electric field $(4I_l/\pi\varepsilon\beta c)(x/a)\ln(\rho/\rho_0)$ at a radius $a/2$ versus observation angle φ for a 1-A beam of radius a. Ratio of beam-pipe radius to beam radius is 100.

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APPENDIX

Here we perform the summation of the radial electric field from all harmonics, starting with Eq. (54). Let us consider the region $r<R$. The summed radial field is given by

$$rE_r=\sum_{n=1}^{\infty}\frac{R\rho_r}{\varepsilon}\sin\left[\frac{n\delta}{2R}\right]\left\{N_n'(n)J_n\left[n\left[1+\frac{x}{R}\right]\right]\times\cos\left[n\frac{\zeta}{R}\right]+J_n'(n)J_n\left[n\left[1+\frac{x}{R}\right]\right]\times\sin\left[n\frac{\zeta}{R}\right]\right\},\quad\text{(A1)}$$

where as before $\zeta=R(\theta-\omega t)$. The derivative terms are well known [10]:

$$J_n'(n)=\frac{2^{2/3}}{3^{1/3}\Gamma(\frac{1}{3})n^{2/3}}=0.410\,85n^{-2/3},$$
$$N_n'(n)=\frac{2^{2/3}3^{1/6}}{\Gamma(\frac{1}{3})n^{2/3}}=0.711\,61n^{-2/3},\quad\text{(A2)}$$

and a suitable expansion for the Bessel functions near the transition (x small) can be found in Ref. [11]. The Hankel functions can be written as

$$H_n^{(1)}\left[n\left[1+\frac{x}{R}\right]\right]=2^{4/3}e^{-i\pi/3}n^{-1/3}\times\mathrm{Ai}\left[2^{1/3}\frac{x}{R}e^{-i\pi/3}n^{2/3}\right],$$
$$H_n^{(2)}\left[n\left[1+\frac{x}{R}\right]\right]=2^{4/3}e^{i\pi/3}n^{-1/3}\times\mathrm{Ai}\left[2^{1/3}\frac{x}{R}e^{i\pi/3}n^{2/3}\right], \quad \text{(A3)}$$

where Ai are the Airy functions. The Bessel functions are given by

$$J_n(z)=\frac{H_n^{(1)}(z)+H_n^{(2)}(z)}{2},$$
$$N_n(z)=\frac{H_n^{(1)}(z)-H_n^{(2)}(z)}{2i}. \quad \text{(A4)}$$

Rewriting Eq. (A1), we find that

$$rE_r=\sum_{n=1}^{\infty}\frac{R\rho_r}{n\varepsilon}\frac{2^2}{3^{1/3}\Gamma(\frac{1}{3})}\sin\left[\frac{n\delta}{2R}\right]\sin\left[n\frac{\zeta}{R}+\frac{\pi}{6}\right]\times\left\{e^{-i\pi/3}\mathrm{Ai}\left[2^{1/3}\frac{x}{R}e^{-i\pi/3}n^{2/3}\right]+e^{i\pi/3}\mathrm{Ai}\left[2^{1/3}\frac{x}{R}e^{i\pi/3}n^{2/3}\right]\right\} \quad \text{(A5)}$$

in this region. Physically we only expect significant contributions for n on the order of R/δ or less. Typically, the beam radius is within an order of magnitude of the bunch length (even for the bunching cases). Note that for significant contributions the Airy function argument is on the order of $n^{-1/3}$ or less. For small arguments, the Airy functions can be well represented by

$$\mathrm{Ai}(e^{\pm i\pi/3}z)=c_1\cos(3^{-1/2}z^{3/2})+e^{\pm i\pi/3}zc_2\cos(6^{-1/2}z^{3/2}), \quad \text{(A6)}$$

where the constants are

$$c_1=\frac{3^{-2/3}}{\Gamma(\frac{2}{3})}=0.355\,02,\qquad c_2=\frac{3^{-1/3}}{\Gamma(\frac{1}{3})}=0.258\,81. \quad \text{(A7)}$$

This expansion leads to this equation for the radial electric field:

$$rE_r=\sum_{n=1}^{\infty}\frac{R\rho_r}{n\varepsilon}\frac{2^2}{3^{1/3}\Gamma(\frac{1}{3})}\sin\left[\frac{n\delta}{2R}\right]\sin\left[n\frac{\zeta}{R}+\frac{\pi}{6}\right]\times\{c_1\cos[\sqrt{\tfrac{2}{3}}(|x/R|)^{3/2}n]+c_2 2^{1/3}\frac{x}{R}n^{2/3}\cos[\sqrt{\tfrac{1}{3}}(|x/R|)^{3/2}n]\}. \quad \text{(A8)}$$

The second term in the expansion for the Airy function can be immediately discarded because its leading term is higher order in x/R.

Using the summations [12]:

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\frac{\pi-x}{2},\quad 0<x<2\pi,$$
$$\sum_{n=1}^{\infty}\frac{\cos nx}{n}=\frac{1}{2}\ln\frac{1}{2(1-\cos x)},\quad 0<x<2\pi, \quad \text{(A9)}$$

we find that the radial electric field for $r<R$ to lowest order is

$$rE_r=\begin{cases}\dfrac{R\rho_r}{2\varepsilon}+\mathcal{T}_{\text{cosine}}, & \sqrt{\tfrac{2}{3}}(|x/R|)^{3/2}<\dfrac{\delta}{2R}-\left|\dfrac{\zeta}{R}\right| \\ \dfrac{R\rho_r}{4\varepsilon}+\mathcal{T}_{\text{cosine}}, & \dfrac{\delta}{2R}-\left|\dfrac{\zeta}{R}\right|<\sqrt{\tfrac{2}{3}}(|x/R|)^{3/2}<\dfrac{\delta}{2R}+\left|\dfrac{\zeta}{R}\right| \\ \mathcal{T}_{\text{cosine}}, & \sqrt{\tfrac{2}{3}}(|x/R|)^{3/2}>\dfrac{\delta}{2R}+\left|\dfrac{\zeta}{R}\right|,\end{cases} \quad \text{(A10)}$$

where the cosine term is given by

$$\mathcal{T}_{\text{cosine}}=\frac{R\rho_r}{\varepsilon}\frac{1}{6\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}\times\ln\frac{\left[\frac{\delta}{2R}+\frac{\zeta}{R}\right]^2-\frac{2}{3}\left|\frac{x}{R}\right|^3}{\left[\frac{\delta}{2R}-\frac{\zeta}{R}\right]^2-\frac{2}{3}\left|\frac{x}{R}\right|^3}. \quad \text{(A11)}$$

The cosine term arises from the imaginary term in Eq. (54), and vanishes if we retained only the Bessel function instead of the outward going Hankel function. For typical bunches, the terms involving x can be ignored, and the cosine term reduces to

$$\mathcal{T}_{\text{cosine}}=\frac{R\rho_r}{2\sqrt{3}\varepsilon\pi}\ln\frac{\delta+2\zeta}{\delta-2\zeta}. \quad \text{(A12)}$$

This term is equal to or bigger than the leading term in Eq. (A10) only for positions within $\delta/200$ of the axial bunch edges.

At the center of the bunch the cosine term vanishes. At radii exceeding the region definitions [Eq. (A10)], the electric field at the center of the bunch consists only of the higher-order fields. For the case of a single electron in a circular orbit, the fields directly transverse to the motion of the electron are always of this form (because the bunch length vanishes). Note from Eq. (A8) we would expect that these fields, although small, would not be symmetric around the electron. This physical effect was observed in direct calculations of the fields of a single electron going around in a bend [13,14].

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